

Business Calculus

Chapter 2 – Applications of Differentiation

Section 2.1 – Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

THE FIRST-DERIVATIVE TEST (FDT) FOR RELATIVE EXTREMA – Pg. 203

1. If $f'(x) < 0$ to the left of *critical value (cv)* and $f'(x) > 0$ to its right, have **relative min.**
2. If $f'(x) > 0$ to the left of *cv* and $f'(x) < 0$ to its right, have **relative max.**
3. If have same sign to the left of *cv* and to its right, neither a relative min nor max.

FIND RELATIVE EXTREMA THEN SKETCH GRAPH – Pg. 206

Note: Endpoints of a closed interval $[a, b]$ can be absolute extrema but *not* relative extrema.

1. Find *critical values (cv's)*:
 - a. Find $f'(x)$
 - b. Set $f'(x) = 0$ to find *cv's*.
 - c. Find where $f'(x)$ is undefined (but $f(x)$ is defined) to get additional *cv's*.
 - i. (Polynomial functions are defined for all real numbers.)
 - d. Find *y*-value for each *cv* by substituting *cv's* into $f(x)$ to use for graphing later.
 - e. Note: *cv's* do not guarantee relative extrema; they are candidates.
2. Use *cv's* to divide *x*-axis number line into intervals.
 - a. Choose a test value in each interval.
3. Substitute test values into $f'(x)$ to determine sign of result (+ or -) in each interval.
 - a. Use FDT criteria to determine any relative extrema.
4. Graph the function:
 - a. Plot points already found in Step 1d above.
 - b. Plot additional points using table of values:
 - i. Choose a number for *x* to find its *y*-value. Or choose a number for *y* to find its *x*-value.
Substitute numbers into $f(x)$
 - ii. Plot enough points to fill out graph.

Section 2.2 – Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

A TEST FOR CONCAVITY – Pg. 216

1. If $f''(x) > 0$, graph is *concave up*. $f'(x)$ is *increasing*, $f(x)$ is *turning up*.
2. If $f''(x) < 0$, graph is *concave down*. $f'(x)$ is *decreasing*, $f(x)$ is *turning down*.

THE SECOND-DERIVATIVE TEST (SDT) FOR RELATIVE EXTREMA – Pg. 218

1. $f(c)$ is a **relative min** if $f''(c) > 0$
2. $f(c)$ is a **relative max** if $f''(c) < 0$
3. If $f''(c) = 0$, use FDT to determine relative extremum.

Note: the *critical value* c is substituted into $f''(x)$ to determine the sign for SDT.

FINDING POINTS OF INFLECTION (POI) – Pg. 221

Note: POI is a change in direction of concavity.

1. Find *possible points of inflection (PPOIs)* at x_0
 - a. Find where $f''(x_0) = 0$ and $f''(x_0)$ *does not exist*.
 - b. Find y-value of PPOI's by substituting x_0 's into $f(x)$ to use for graphing later.
2. Determine whether the PPOIs are actually POIs:
 - a. Use x_0 's to divide x-axis number line into intervals.
 - b. Choose a test value in each interval.
 - c. Substitute test values into $f''(x)$ to determine sign of result (+ or -) in each interval.
 - i. If sign is different to the left of x_0 than to its right, have **POI**.
 - ii. If sign is same to the left of x_0 as to its right, no POI.
 - iii. Find y-value of POI's by substituting x_0 's into $f(x)$ to use for graphing later.

STRATEGY FOR SKETCHING GRAPHS – POLYNOMIAL FUNCTIONS – Pg. 222

1. *Derivatives and domain:*

- Find $f'(x)$
- Find $f''(x)$
- Find domain of $f(x)$. (Domain of polynomial functions is all real numbers.)

2. *Critical values (cv's) of $f(x)$*

- Set $f'(x) = 0$ to find cv's.
- Find where $f'(x)$ is undefined (but $f(x)$ is defined) to get additional cv's.
 - (Polynomial functions are defined for all real numbers.)
- Find y-value for each cv by substituting cv's into $f(x)$ to use for graphing later.
- Note: cv's do not guarantee relative extrema; they are candidates.

3. *Increasing / decreasing; and relative extrema:*

- Substitute cv's into $f''(x)$ for SDT.
- If $f''(c) < 0$, have **relative max**
 - $f(x)$ is *increasing* to the left of cv and *decreasing* to its right. List interval.
- If $f''(c) > 0$, have **relative min**
 - $f(x)$ is *decreasing* to the left of cv and *increasing* to its right. List interval.
- If $f''(c) = 0$, must use FDT to see if have relative extrema.

4. *Points of Inflection (POIs):*

Note: POI is a change in direction of concavity.

- Find *possible points of inflection (PPOIs)* at x_0
 - Find where $f''(x_0) = 0$ and $f''(x_0)$ does not exist.
 - Find y-value of PPOI's by substituting x_0 's into $f(x)$ to use for graphing later.
- Determine whether the PPOIs are actually POIs:
 - Use x_0 's to divide x-axis number line into intervals.
 - Choose a test value in each interval.
 - Substitute test values into $f''(x)$ to determine sign of result (+ or -) in each interval.
 - If sign is different to the left of x_0 than to its right, have **POI**.
 - If sign is same to the left of x_0 as to its right, no POI.
 - Find y-value of POI's by substituting x_0 's into $f(x)$ to use for graphing later.

5. *Concavity:*

- List intervals of concavity using the PPOI x-axis number line in Step 4.
- If result in Step 4 was:
 - $f''(x) > 0$, graph is *concave up* on that PPOI interval.
 - $f''(x) < 0$, graph is *concave down* on that PPOI interval.

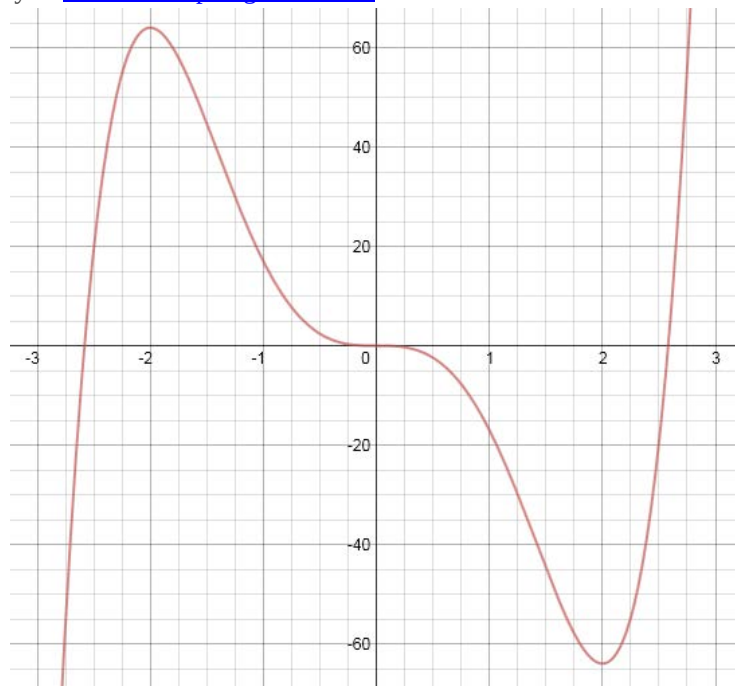
6. *Sketch the graph:*

- Plot points already found in Step 2 and Step 4 above.
- Plot additional points using table of values:
 - Choose a number for x to find its y-value. Or choose a number for y to find its x-value.
Substitute numbers into $f(x)$
 - Plot enough points to fill out graph.
- Use the **intervals table** on next page to help sketch graphs of polynomial functions.

Example: The polynomial function below is sketched using the **intervals table** below.

$$f(x) = 3x^5 - 20x^3$$

Graph below courtesy of [Desmos Graphing Calculator](#)



Intervals Table for Sketching Graphs – Polynomial Functions (Example)

Interval Line	$\leftarrow \begin{array}{c} \approx -2.6 \quad -2 \quad \approx -1.4 \quad 0 \quad \approx +1.4 \quad +2 \quad \approx +2.6 \\ (-\infty, \approx -2.6) \quad (\approx -2.6, -2) \quad (-2, \approx -1.4) \quad (\approx -1.4, 0) \quad (0, \approx +1.4) \quad (\approx +1.4, +2) \quad (+2, \approx +2.6) \quad (\approx +2.6, +\infty) \end{array} \rightarrow$							
Test Values x	$x = -3$	$x = -2.5$	$x = -1.5$	$x = -1$	$x = +1$	$x = +1.5$	$x = +2.5$	$x = +3$
$f'(x)$ Sign	+	+	-	-	-	-	+	+
$f''(x)$ Sign	-	-	-	+	-	+	+	+
Increasing / Decreasing	Increasing	Increasing	Decreasing	Decreasing	Decreasing	Decreasing	Increasing	Increasing
Concave up / Down	Concave Down	Concave Down	Concave Down	Concave Up	Concave Down	Concave Up	Concave Up	Concave Up
Graph Shape								

Notes:

- The values in the table above are from the example problem. Notice that the last row of table, Graph Shape, agrees with the graph image for all intervals.
- Partition the interval line with x -values from: x -intercepts, critical values, PPOI's.
 - List the x -values in numerical order from left to right.
 - In example above, the x -values are:
 - x -intercepts: $\approx -2.6, \approx +2.6$
 - Critical values: $-2, +2$
 - PPOI's: $\approx -1.4, 0, \approx +1.4$
- Substitute **test values x** into $f'(x)$ and $f''(x)$ to calculate the signs.
 - Then based on the signs, determine if function is increasing/decreasing and concavity.
 - Finally for each interval, sketch a graph shape based on increasing/decreasing and concavity.

Section 2.3 – Graph Sketching: Asymptotes and Rational Functions

VERTICAL ASYMPTOTE (VA) – Pg. 235

$$\lim_{x \rightarrow a^-} f(x) = \infty$$

$$\lim_{x \rightarrow a^-} f(x) = -\infty$$

$$\lim_{x \rightarrow a^+} f(x) = \infty$$

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

Notes:

- A rational function may or may not have VA. Will not have VA if denominator will never be 0.
- If original rational function is already simplified (cannot factor), then if a makes denominator 0, have VA at $x = a$.
- If factored and cancelled a factor of the rational function, there is no VA at a that made denominator 0. Instead, it is called a “point of discontinuity,” a “hole.”
- Graph of rational function *never* crosses VA.

HORIZONTAL ASYMPTOTE (HA) – Pg. 237

$$\lim_{x \rightarrow -\infty} f(x) = b$$

$$\lim_{x \rightarrow \infty} f(x) = b$$

Notes:

- HA occurs when degree of numerator $\leq$ degree of denominator.
 - When degree of numerator = degree of denominator, HA is the leading coefficients of numerator and denominator.
 - When degree of numerator < degree of denominator, HA is the x-axis ($y = 0$).
- Graph of rational function *may or may not* cross an HA.
- Graph of rational function can have at most 2 HA's.

SLANT ASYMPTOTE (OBLIQUE ASYMPTOTE) – Pg. 239

Notes:

- Slant Asymptote occurs when degree of numerator is *exactly 1 more* than the degree of denominator.
- To find the *linear* slant asymptote, divide the numerator by the denominator using long division. The quotient is the slant asymptote.
- Graph of rational function *can* cross a slant asymptote.
- A rational function will have either an HA or a slant asymptote, but *not both*.

INTERCEPTS – Pg. 240

Notes:

- **x-Intercepts:**
 - To find x-intercepts of rational function, set $f(x) = 0$.
 - The x-intercepts occur when the numerator is zero and the denominator is not zero.
 - Factor numerator and denominator.
 - Set numerator = 0 and solve for x.
 - Check that these x-values do not make the denominator 0 also. If any do, they are not x-intercepts.
- **y-Intercept:**
 - To find y-intercept of rational function, set $f(0)$.

STRATEGY FOR SKETCHING GRAPHS – RATIONAL FUNCTIONS – Pg. 241

1. *Intercepts:*
 - a. Find the x-intercepts and the y-intercept.
2. *Asymptotes:*
 - a. Find any VA's, HA's, or slant asymptotes.
3. *Derivatives and domain:*
 - a. Find $f'(x)$
 - b. Find $f''(x)$
 - c. Find domain of $f(x)$. List x-values that make denominator = 0 (from VA step).
4. *Critical values (cv's) of $f(x)$*
 - a. Set $f'(x) = 0$ to find cv's. Set numerator = 0.
 - b. Find where $f'(x)$ is undefined (but $f(x)$ is defined) to get additional cv's.
 - c. Find y-value for each cv by substituting cv's into $f(x)$ to use for graphing later.
 - d. Note: cv's do not guarantee relative extrema; they are candidates.
5. *Increasing / decreasing; and relative extrema:*
 - a. Substitute cv's into $f''(x)$ for SDT.
 - b. If $f''(c) < 0$, have **relative max**
 - i. $f(x)$ is *increasing* to the left of cv and *decreasing* to its right. List interval.
 - c. If $f''(c) > 0$, have **relative min**
 - i. $f(x)$ is *decreasing* to the left of cv and *increasing* to its right. List interval.
 - d. If $f''(c) = 0$, must use FDT to see if have relative extrema.
6. *Points of Inflection (POIs):*

Note: POI is a change in direction of concavity.

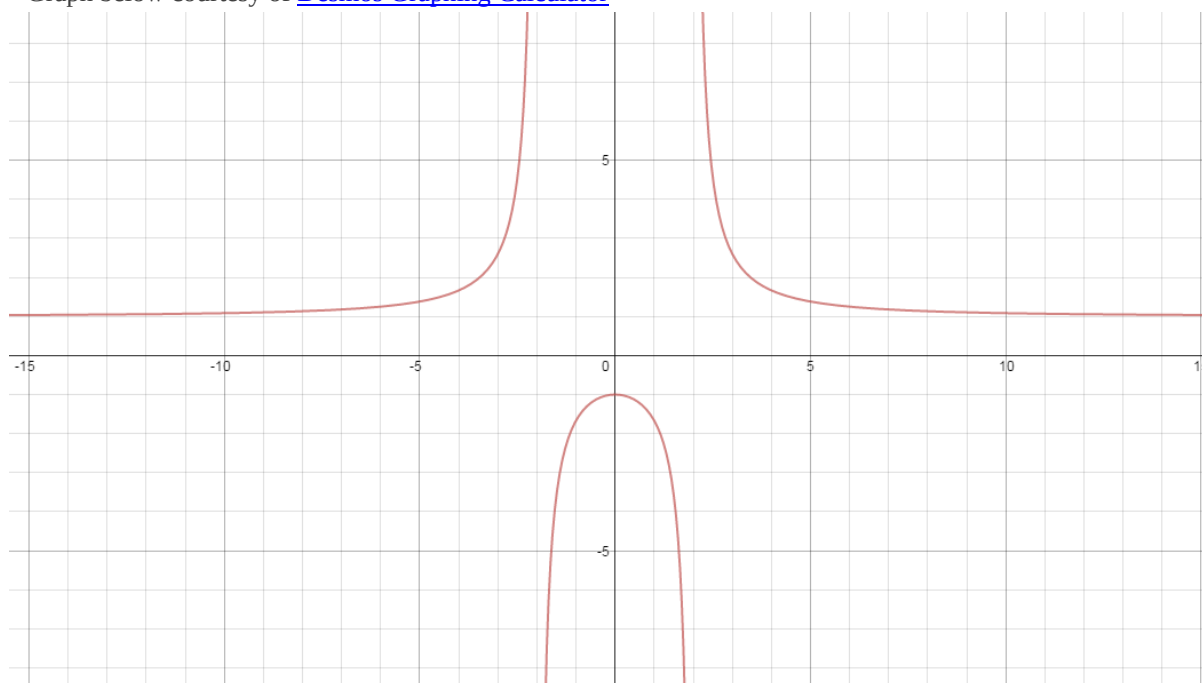
 - a. Find *possible points of inflection (PPOIs)* at x_0
 - i. Find where $f''(x_0) = 0$ and $f''(x_0)$ does not exist.
 1. If x_0 does not exist at $f(x)$, not a PPOI.
 - ii. Find y-value of PPOI's by substituting x_0 's into $f(x)$ to use for graphing later.
 - b. Determine whether the PPOIs are actually POIs:
 - i. Use x_0 's to divide x-axis number line into intervals.
 - ii. Choose a test value in each interval.
 - iii. Substitute test values into $f''(x)$ to determine sign of result (+ or -) in each interval.
 1. If sign is different to the left of x_0 than to its right, have **POI**.
 2. If sign is same to the left of x_0 as to its right, no POI.
 3. Find y-value of POI's by substituting x_0 's into $f(x)$ to use for graphing later.
7. *Concavity:*
 - a. List intervals of concavity using the PPOI x-axis number line in Step 6.
 - b. If result in Step 6 was:
 - i. $f''(x) > 0$, graph is *concave up* on that PPOI interval.
 - ii. $f''(x) < 0$, graph is *concave down* on that PPOI interval.
8. *Sketch the graph:*

- a. Plot points already found in Step 4 and Step 6 above.
- b. Plot additional points using table of values:
 - i. Choose a number for x to find its y -value. Or choose a number for y to find its x -value. Substitute numbers into $f(x)$
 - ii. Plot enough points to fill out graph.
- c. Use the **intervals table** on next page to help sketch graphs of rational functions.

Example: The rational function below is sketched using the **intervals table** on next page.

$$f(x) = \frac{x^2 + 4}{x^2 - 4}$$

Graph below courtesy of [Desmos Graphing Calculator](#)



Intervals Table for Sketching Graphs – Rational Functions (Example)

Interval Line	$\leftarrow \begin{array}{c} \quad \quad \quad \\ -2 \quad 0 \quad +2 \end{array} \rightarrow$ $(-\infty, -2) \quad (-2, 0) \quad (0, +2) \quad (+2, +\infty)$			
Test Values x	$x = -10$	$x = -1$	$x = +1$	$x = +10$
$f(x)$ Value	$+1.08 > +1$ (above HA)	--	--	$+1.08 > +1$ (above HA)
$f'(x)$ Sign	$\frac{+}{+} = +$	$\frac{+}{+} = +$	$\frac{-}{+} = -$	$\frac{-}{+} = -$
$f''(x)$ Sign	$\frac{+}{+} = +$	$\frac{+}{-} = -$	$\frac{+}{-} = -$	$\frac{+}{+} = +$
Increasing / Decreasing	Increasing	Increasing	Decreasing	Decreasing
Concave up / Down	Concave Up	Concave Down	Concave Down	Concave Up
Graph Shape				

Notes:

- The values in the table are from the example problem on previous page. Notice that the last row of table, Graph Shape, agrees with the graph image for all intervals.
- Partition the interval line with x -values from: x -intercepts, VA's, critical values, PPOI's.
 - List the x -values in numerical order from left to right.
 - In example above, the x -values are -2 (VA), 0 (cv), and $+2$ (VA).
- The $f(x)$ Value row is used to determine the end behavior of the horizontal asymptote (HA).
 - The test values of $x = -10$ and $x = +10$ both result in the $f(x)$ value of $+1.08$, which is greater than the HA of $y = 1$. This means that the end behavior of the rational function places the graph above the HA at both ends on the x -axis.
 - Choose fairly large x -values to test for end behavior of HA.
 - The "inside" intervals for $f(x)$ do not need to be computed and have "--" in the table cell.
- Substitute test values x into $f'(x)$ and $f''(x)$ to calculate the signs.
 - Then based on the signs, determine if function is increasing/decreasing and concavity.
 - Finally for each interval, sketch a graph shape based on increasing/decreasing and concavity.
 - Note: This $\frac{-}{+} = -$ indicates the sign of the numerator, denominator, and the overall sign.
- To determine end behavior of a vertical asymptote, take the limit to the left and to the right of the VA.
 - If the limit is negative, the graph is decreasing on that side of the VA.
 - If the limit is positive, the graph is increasing on that side of the VA.
 - Note: This is not shown in the table above.

Section 2.4 – Using Derivatives to Find Absolute Maximum and Minimum Values

MAXIMUM-MINIMUM PRINCIPLE 1 – Pg. 251

For a *closed* interval $[a, b]$.

1. Find $f'(x)$
2. Find *critical values* (cv's) in $[a, b]$
 - a. Find where $f'(c) = 0$ and $f'(c)$ *does not exist*.
 - b. A cv must be within interval $[a, b]$ to be considered for a max or min.
3. List values from Step 2 and the endpoints of the interval $[a, b]$.
4. Evaluate $f(x)$ for each value in Step 3.
 - a. Largest value is **absolute max**
 - b. Smallest value is **absolute min**

Notes:

- Absolute max and min values *can* occur at more than one point.
- Endpoints of a closed interval $[a, b]$ can be absolute extrema but *not* relative extrema.

MAXIMUM-MINIMUM PRINCIPLE 2 – Pg. 253

For *exactly one* cv in interval I .

1. Find $f'(x)$
 - a. $f'(x) = 0$ produces exactly one cv.
2. Find $f''(x)$
 - a. If $f''(c) < 0$, $f(c)$ is an **absolute max**
 - b. If $f''(c) > 0$, $f(c)$ is a **relative min**
 - c. If $f''(c) = 0$, use Maximum-Minimum Principle 1; or we must know more about the behavior of the function to declare an absolute max or min.

Notes:

- The *critical value* c is substituted into $f''(x)$ to determine the sign for SDT.
- If the endpoints are not listed, they are considered to be $(-\infty, \infty)$.

A STRATEGY FOR FINDING ABSOLUTE MAXIMUM AND MINIMUM VALUES – Pg. 255

1. Find $f'(x)$
2. Find critical values.
 - a. Find where $f'(c) = 0$ and $f'(c)$ does not exist.
3. If closed interval $[a, b]$ and:
 - a. There is *more than one* critical value, use Maximum-Minimum Principle 1.
 - b. There is *exactly one* critical value, use either Maximum-Minimum Principle 1 or Maximum-Minimum Principle 2.
 - i. If it is easy to find $f''(x)$, use Maximum-Minimum Principle 2.
4. If open interval $(-\infty, \infty)$, $(0, \infty)$, or (a, b) and:
 - a. There is *only one* critical value, use Maximum-Minimum Principle 2.
 - i. Result will either be a max or min.
 - b. There is *more than one* critical value, beyond scope of this course.

Section 2.5 – Maximum-Minimum Problems; Business and Economics Applications

A STRATEGY FOR SOLVING MAXIMUM-MINIMUM PROBLEMS – (Pg. 263)

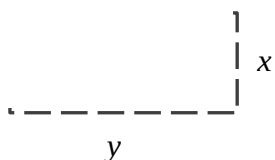
1. Read problem carefully. Make a drawing if it makes sense to do so.
2. List variables and constants. Note unit of measure. Label measurements on drawing.
3. Translate verbiage to an *equation* with the quantity Q to be minimized or maximized. Represent Q in terms of the variables listed in Step 2.
4. Express Q as a function of one variable. Determine max or min values and the points where they occur.

EX: 1 MAXIMIZE AREA – (Pg. 262)

Problem: Rectangular area needs to be fenced off only at 2 adjacent sides and maximized.

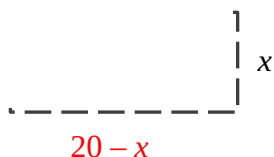
STEP 1 – “*Make a drawing.*”

- Let x be the dimensions of one side; (*width* or smaller side).
- Let y be the dimensions of other side, (*length* or larger side).



STEP 2 – “List variables and constants, unit of measure. Label measurements on drawing.”

- Have 20 feet of fencing available.
- Since $x + y = 20$, solve for y to have the 2 dimensions in terms of just one variable, x . So the length (larger side) becomes $y = 20 - x$. Relabel the y side of drawing as $20 - x$.



STEP 3 – “Translate to an equation with the quantity Q to be maximized. Represent Q in terms of the variables listed in Step 2.”

- Since trying to find maximum area, need to use the Area Formula: $A = lw$
- Substitute problem variables ($20 - x$ and x) into Area Formula:
 $A = lw$ – Area Formula is the *starting equation*
 $A = (20 - x)x$ – Area Formula after substituting problem variables

STEP 4 – “Express Q as a function of one variable. Determine max or min values and the points where they occur.”

$$A(x) = 20x - x^2 \quad \begin{array}{l} \text{– Area Formula to be maximized, after multiplying and simplifying.} \\ \text{– Domain is } (0, 20) \text{ since dimensions in feet cannot be } < 0, 0, 20, \text{ or } > 20. \end{array}$$

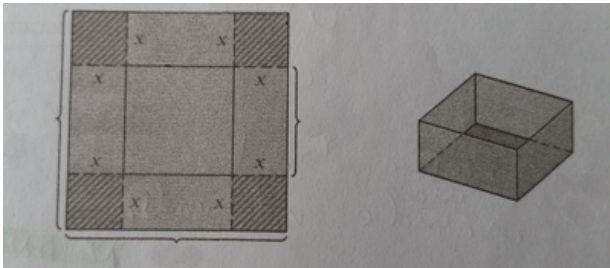
- Find $A'(x) = 20 - 2x$
- Set $A'(x) = 0$ to find *critical values* cv 's.
 - o $20 - 2x = 0$
 - o $x = 10$
- Since there's only 1 cv , use *Maximum-Minimum Principle 2* (second derivative) to determine if this cv is a maximum.
 - o $A''(x) = -2$ – This is a negative constant so the cv $x = 10$ is a maximum.
- Conclusion:
 - o $x = 10$ represents the dimension of 10 feet for side x .
 - o Substitute 10 for x into $y = 20 - x \Rightarrow y = 20 - 10 \Rightarrow y = 10$ which represents the dimension of 10 feet for side y .
 - o Then find area by $A = lw \Rightarrow A = (10)(10) \Rightarrow A = 100$
 - o The dimensions of the two sides should each be 10 feet to give a maximum area of 100ft^2 .
- Video Related To This Example: <https://www.youtube.com/watch?v=jpFquCszOX0>

EX: 2 MAXIMIZE VOLUME – (Pg. 264)

Problem: A cardboard 8 in. by 8 in. needs to have a square cut out from corners so the sides can be folded up to make an open-top box. What dimension will yield maximum volume? What is the maximum volume?

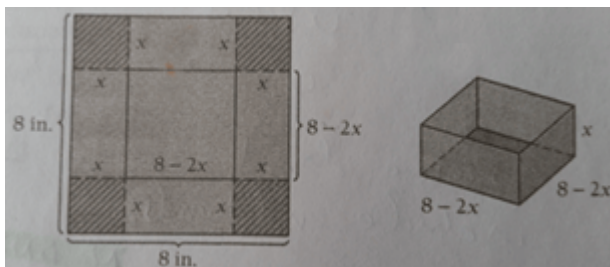
STEP 1 – “*Make a drawing.*”

- Let x be the dimensions of each square corner to be cut.



STEP 2 – “*List variables and constants, unit of measure. Label measurements on drawing.*”

- Since original square is 8 in. by 8 in., after the four corner squares (x 's) are cut out, the dimensions of the sides will be $(8 - 2x)$ in. by $(8 - 2x)$ in. The part labeled x is going to be the height of the box after the sides are folded up.



STEP 3 – “*Translate to an equation with the quantity Q to be maximized. Represent Q in terms of the variables listed in Step 2.*”

- Since trying to find maximum volume, need to use the Volume Formula: $V = lwh$
- Substitute problem variables of length, width, height $(8 - 2x)$, $(8 - 2x)$, x into Volume Formula:
 $V = lwh$ – Volume Formula is the *starting equation*
 $V = (8 - 2x)(8 - 2x)x$ – Volume Formula after substituting problem variables

STEP 4 – “Express Q as a function of one variable. Determine max or min values and the points where they occur.”

$$V(x) = 4x^3 - 32x^2 + 64x$$

– Volume Formula to be maximized, after multiplying and simplifying.
 – Domain: since a side $8 - 2x > 0$, each square corner $x < 4$.
 Therefore $(0, 4)$.

- Find $V'(x) = 12x^2 - 64x + 64$
- Set $V'(x) = 0$ to find *critical values* cv 's.
 - o $12x^2 - 64x + 64 = 0$
 - o $x = \frac{4}{3}, 4$
- Because the cv 4 is not in the domain $(0, 4)$, have only one cv , $\frac{4}{3}$.
- Since there's only 1 cv , use *Maximum-Minimum Principle 2* (second derivative) to determine if this cv is a maximum.
 - o $V''(x) = 24x - 64$ – Find second derivative.
 - o $V''\left(\frac{4}{3}\right) = 24\left(\frac{4}{3}\right) - 64 \Rightarrow 32 - 64 < 0$ – Substitute $\frac{4}{3}$ into V'' and result is negative, therefore $\frac{4}{3}$ is a maximum.
- Conclusion:
 - o $x = \frac{4}{3}$ represents the $\frac{4}{3}$ in. of the corner square, x , that needs to be cut out to maximize the box's volume.
 - o After the $\frac{4}{3}$ in. has been cut out from each corner and the sides have been folded up, the dimensions of the two equal sides of the box will be: $8 - 2x \Rightarrow 8 - 2\left(\frac{4}{3}\right) \Rightarrow 5\frac{1}{3}$ in.
 - o The height of the box is $\frac{4}{3}$ in. which is obtained when the x is cut out (the $\frac{4}{3}$ in.) and the remaining part of the side is folded up (the $5\frac{1}{3}$ in.).
 - o The maximum volume of the box is:

$$V(x) = 4x^3 - 32x^2 + 64x \Rightarrow V\left(\frac{4}{3}\right) = 4\left(\frac{4}{3}\right)^3 - 32\left(\frac{4}{3}\right)^2 + 64\left(\frac{4}{3}\right) \Rightarrow 37\frac{25}{27} in^3$$
- Video Related To This Example: <https://www.youtube.com/watch?v=wCVuupUndb0>

Ex: 3 Minimize Surface Area (Pg. 265) – <https://www.youtube.com/watch?v=T1m8BN7UbOg>

Ex: 4 Maximize Revenue and Profit (Pg. 267) – <https://www.youtube.com/watch?v=-TzmLSQZFoI>

EXERCISE SET 2.5, # 31: MAXIMIZE REVENUE (*Determining a Ticket Price*) – (Pg. 274)

Similar to **Ex: 5 Maximize Revenue** (Pg. 268)

Problem: A university is trying to determine what price to charge for tickets to football games. At a price of \$18 per ticket, attendance averages 40,000 people per game. Every decrease of \$3 adds 10,000 people to the average number. Every person at the game spends an average of \$4.50 on concessions. What price per ticket should be charged in order to maximize revenue? How many people will attend at that price?

STEP 1 – “Make a drawing.”

- Not applicable.

STEP 2 – “List variables and constants, unit of measure.”

- Revenue = $R(x) = \text{Quantity} \cdot \text{Price}$
- Two sources of revenue in this type of problem: ticket revenue + concession revenue.
- Ticket revenue:
 - o Ticket revenue with “normal” operation (average attendance with normal ticket price):
 - Quantity = 40,000 people
 - Price = \$18 ticket price
 - $R(x) = (40,000 \text{ people}) (\$18)$
 - o Ticket revenue with problem constraints (restrictions on “normal” operation):
 - Quantity = 40,000 people + 10,000 people (for every decrease of \$3 per ticket)
 - $(40,000 + 10,000x)$
 - Note: The x is the number of times the ticket is reduced by \$3. The x represents some multiple of the number of times the ticket is reduced by \$3. The x is a multiplier with the change in attendance.
 - Price = \$18 and each occurrence that the ticket is decreased by \$3.
 - $(18 - 3x)$
 - Note: As noted above, x is the number of times the ticket is reduced by \$3. The x represents some multiple of the number of times the ticket is reduced by \$3.
- Concession revenue:
 - o Concession revenue with “normal” operation (average attendance with normal concession spending):
 - Quantity = 40,000 people
 - Price = \$4.50 concession spending per person
 - $R(x) = (40,000 \text{ people}) (\$4.50)$
 - o Concession revenue with problem constraints (restrictions on “normal” operation):
 - Quantity = 40,000 people + 10,000 people (for every decrease of \$3 per ticket)
 - Price = \$4.50 concession spending per person *remains the same as “normal” operation*. The \$4.50 is a constant.
 - $(40,000 + 10,000x) (4.50)$
 - Note: The quantity “attendance” $(40,000 + 10,000x)$ is the same for concession revenue as it is for ticket revenue.

STEP 3 – “Translate to an equation with the quantity $R(x)$ to be maximized. Represent $R(x)$ in terms of the variables listed in Step 2.”

Find total revenue function:

$$R(x) = \text{Total revenue} = (\text{Ticket revenue}) + (\text{Concession revenue})$$

$$R(x) = (\text{Number of people}) (\text{Ticket price}) + (\text{Number of people}) (\text{Concession spending})$$

$$R(x) = (40,000 + 10,000x) (18 - 3x) + (40,000 + 10,000x) (4.50)$$

$$R(x) = 720,000 - 120,000x + 180,000x - 30,000x^2 + 180,000 + 45,000x$$

$$R(x) = -30,000x^2 + 105,000x + 900,000$$

$$\text{Find } R'(x) = -60,000x + 105,000$$

Set $R'(x) = 0$ to find *critical values* cv 's.

$$-60,000x + 105,000 = 0$$

$$x = 1.75$$

Since there's only 1 cv , use *Maximum-Minimum Principle 2* (second derivative) to determine if this cv is a maximum.

$$R''(x) = -60,000$$

– Since second derivative is a negative constant, 1.75 is a maximum.

- Conclusion:

- o Price per ticket to charge:

$$(18 - 3x)$$

$$(18 - 3 \cdot 1.75)$$

$$12.75$$

– Substitute 1.75 for x

– To maximize profit, the ticket price should be \$12.75

- o Attendance:

$$(40,000 + 10,000x)$$

$$(40,000 + 10,000 \cdot 1.75)$$

$$57,000$$

– Substitute 1.75 for x

– At a ticket price of \$12.75, the amount of people that will attend per game will be 57,000.

Additional Resources for This Type of Problem

- Videos:

- o https://www.youtube.com/watch?v=vfIFLryA_DU

- o <https://www.youtube.com/watch?v=WWkwunkQc-c>

Chapter 4 – Integration

Section 4.1 – Antidifferentiation

RULES OF ANTIDIFFERENTIATION – Pg. 391

Constant Rule

$$\int k dx = kx + C$$

Power Rule

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1$$

Natural Logarithm Rule

$$\int \frac{1}{x} dx = \ln x + C, \quad x > 0$$

Exponential Rule (base e)

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C, \quad a \neq 0$$

PROPERTIES OF ANTIDIFFERENTIATION – Pg. 393

$$\int [cf(x)] dx = c \int f(x) dx$$

Section 4.2 – Antiderivatives as Areas

RIEMANN SUMMATION – Pg. 401-405

Area of Region Under the Curve

$$\sum_{i=1}^n f(x_i)\Delta x, \quad \text{where } \Delta x = \frac{b-a}{n}$$

STEPS FOR THE PROCESS OF RIEMANN SUMMATION – Pg. 405

1. Draw the graph of $f(x)$.
2. Subdivide the interval $[a, b]$ into n subintervals of equal width. Calculate the *width* of each rectangle by using the formula:

$$\Delta x = \frac{b-a}{n}$$

3. Construct rectangles above the subintervals such that the *top left corner* of each rectangle touches the graph.
4. Determine the *area of each rectangle*.
5. *Sum these areas* to arrive at an approximation for the total area under the curve.

DEFINITE INTEGRAL – Pg. 405

A *definite integral* is the limit as $n \rightarrow \infty$ (equivalently, $\Delta x \rightarrow 0$) of the Riemann sum.

$$\text{Exact Area} = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n f(x_i)\Delta x = \int_a^b f(x)dx$$

Section 4.3 – Area and Definite Integrals

DEFINITE INTEGRAL – Pg. 414

$$\int_a^b f(x)dx = F(b) - F(a)$$

Where a is lower limit and b is higher limit.

THE FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS – Pg. 415

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x = \int_a^b f(x)dx = F(b) - F(a)$$

Section 4.4 – Properties of Definite Integrals

ADDITIVE PROPERTY OF DEFINITE INTEGRALS – Pg. 425

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

For $a < b < c$

Useful for *piecewise* functions.

AREA OF REGION BOUNDED BY TWO GRAPHS – Pg. 428

$$\int_a^b [f(x) - g(x)]dx$$

Where $f(x) \geq g(x)$ over the interval $[a, b]$

STEPS TO FIND AREA OF REGION BOUNDED BY TWO GRAPHS

1. Sketch graphs to determine which one is the upper graph.
2. Set $f(x) = g(x)$ and solve for x . These x -values are the limits of integration.
3. Set up integral by substituting the appropriate functions as the $f(x)$ and $g(x)$ into the formula above.
4. Simplify $f(x) - g(x)$ to get one polynomial.
5. Integrate.

AVERAGE VALUE OF A FUNCTION – Pg. 431

$$y_{av} = \frac{1}{b-a} \int_a^b f(x)dx$$

Section 4.5 – Integration Techniques: Substitution

Power Rule

$$\int u^r du = \frac{u^{r+1}}{r+1} + C, \quad r \neq -1$$

Exponential Rule (base e)

$$\int e^u du = e^u + C$$

Natural Logarithm Rule

$$\int \frac{1}{u} du = \ln|u| + C, \quad \text{or} \quad \int \frac{1}{u} du = \ln u + C, \quad \text{if } u > 0$$

STRATEGY FOR SUBSTITUTION – Pg. 442

1. Decide which rule is appropriate; see above.
 - a. If *Power Rule*, let u be the base.
 - b. If *Exponential Rule (base e)*, let u be the expression in the exponent.
 - c. If *Natural Logarithm Rule*, let u be the denominator.
2. Determine du .
3. Ensure substitution accounts for all factors in integrand. May need to insert constants or make an extra substitution.
4. Integrate.
5. Reverse the substitution. If there are bounds, evaluate integral *after* substitution has been reversed.
6. Check by differentiating.

Chapter 5 – Applications of Integration

Section 5.1 – An Economics Application: Consumer Surplus and Producer Surplus

CONSUMER SURPLUS (DEMAND CURVE) – Pg. 475

Consumer surplus is defined for the point (Q, P) as

$$\int_0^Q D(x)dx - QP$$

Where,

Q is *Quantity*

P is *Price*

$D(x)$ is *total utility*

QP is *total cost*

Consumer surplus is the “pleasure received but did not pay for.” Aka *Utility* or U .

PRODUCER SURPLUS (SUPPLY CURVE) – Pg. 476

Producer surplus is defined for the point (Q, P) as

$$QP - \int_0^Q S(x)dx$$

Where,

Q is *Quantity*

P is *Price*

QP is *revenue*

$S(x)$ is *total cost*

Producer surplus is a contribution to profit.

EQUILIBRIUM POINT (x_E, p_E) – Pg. 477

$$D(x) = S(x)$$

Where,

$$x_E = Q$$

$$p_E = P$$

Solve for x then substitute into either $D(x)$ or $S(x)$ to find y .

CONSUMER SURPLUS AT THE EQUILIBRIUM POINT – Pg. 478

$$\int_0^{x_E} D(x)dx - x_E p_E$$

Where,

$D(x)$ is *total utility*

$x_E p_E$ is *total cost*

PRODUCER SURPLUS AT THE EQUILIBRIUM POINT – Pg. 478

$$x_E p_E - \int_0^{x_E} S(x)dx$$

Where,

$x_E p_E$ is *revenue*

$S(x)$ is *total cost*